# What is a number?

*Abstract:*
I see the major problem for the teaching of numbers in school as the conflicting number concepts the students implicitly are offered through their school-education.
I think it must be a natural problem for POME to discuss, because a useful solution must combine point of views from philosophy of mathematics with educational questions.

In my paper I will more theoretically try to discuss reasons for these 'conceptual breaks' and suggest some practical answers.
My actual suggestion for an operative 'school-definition' is something like: A number is a symbol, which specify a position relative to the unit.
This is implicitly build on a geometrically understanding with numbers related to a number line or a number plane. Without this geometrically foundation but then also less operative we could define as a reasonable guiding for the schoolteacher: Numbers are a structured description system based on a unit.

*Introduction:*
I see the major problem for the teaching of numbers in school as the conflicting number concepts the students implicitly are offered through their school-education.
E. Lehtinen spoke at icme-10 in Copenhagen 2004 about such problems:
Cited from the abstract: *'Thus, every extension of the number concept demands new rules to be learned for operations and the use of a new kind of logic often leading to many different, but systematic problems and misconceptions in mathematics learning'*
In the discussion afterwards he especially pointed at two major conceptual conflicts connected to fractions and to the real numbers.
I will more theoretically try to discuss reasons for these 'conceptual breaks' and then suggest some practical answer.  I will start with a short discussion about 'ideas about mathematics' by dividing the discussion in three parts: The relations to 'the real world', the discussion of how to make and ensure correctly deductions and thirdly the discussion about understanding mathematics.
Next I will discuss number concepts relating to 'the three parts' and try to explain why the concepts are 'breaking' and finally I will see on some possible answers.

*<u>Ideas about mathematics:</u>*

Aristoteles characterize the methodology of demonstrative or deductive sciences by three postulates of deduction, self-evidence and reality. [See for example Beth - Piaget 1966, p 36 - 37].

Also to day I think it can be useful to divide the discussion of mathematics in three parts: The more technical discussion of how to make and ensure correctly deductions, the more individually discussion about understanding mathematics and thirdly the relations to 'the real world'.

*The relations to 'the real world':*

In stead of the idea about a true relation between theories and 'the real world', we normally to day have a more constructivistic attitude, where we make more or less useful models of the world.

This change from 'true' to 'useful' can be understood as a consequence of a fundamental limit for our knowledge - for example Ernst von Glasersfeld (1995, p. 51) formulated as one of the fundamental principles for radical constructivism:

*'- The function of cognition is adaptive, in the biological sense of the term, tending towards fit or viability; - cognition serves the subject's organization of the experiential world, not the discovery of an objective ontological reality.'*

Or it can be accepted as a practical limit - for example the concrete problems by making reasonable predictions by models for the weather or economy etc.

It is naturally to connect the mathematics we can use to build models of parts of the world - or as for example pointed out by Ole Skovsmose (1990, p 135 - 142) - also models of hypothetical worlds - with languages. Models are used for description, prediction and communication both in special situations and in our daily social life. Pointing at the existence of a social reality Paul Ernest (19xx) write:

*'Mathematics is the theory of form and structure that arises within language.'*

Sub-conclusion: In school-education the connection to physical and social reality is very important implying a good deal of weight on the language side of mathematics.

*'Inside' the mathematical building:*

The way we ensure the correctness of deductions has been developed, but already Euclid gives in the famous 'Elements' a master example of how a deductive description of mathematics could bee

carried out in practice. The deductions starting with some principles and are divided in small steps. To day the deductions can normally be brought on a form, where we have a chain of signs, changing after some algorithmic described principles, so it is possible with a mechanically (computer) control of the proofs. This ensure a highly degree of correctness. Of course, there could still be mistakes, but it will be a great surprise if it happens in a 'simple' proof controlled many times by many people.
I see formalism, logicism and structuralism (Bourbaki) as especially concerning about the 'inside' of the mathematical building. But of course - some as for example Bertrand Russell - try to make connection to the 'real world'.

*Understanding mathematics:*

The very formally descriptions of the mathematical theories rise the question: How to give the signs some meanings? We can make a program to a computer, so we can create al possible combinations of letters and blanks.  After some time it should has produced al poems with less than 100 signs. But who should know when the beautiful poem arise? A lot of signs - is just a lot of signs. Someone has to make an interpretation of the signs (or try to build this into a program).
I transform the 'self-evidence' to the individual acceptance and understanding of theories.
I see the intuitionism and constructions of Descartes and Kant (and others) (Beth - Piaget 1966, Gundersen (199x)) also as discussions of how to connect the human mind with mathematics? If our brain is God-given then this maybe can give us a to true understanding of the real world, but else intuition could both relate to our biologically fit to the physical and social world and our earlier constructions as children as for example Piaget is investigating.
If we discuss the understanding of signs in a novel - for example novels about Harry Potter or the 'Lord of the Rings' we often understand the signs by building up a fantasy world inside our brains. The world is an individual construction related to the novel, but with the kind of existence, that we can dream and fantasize inside it.  When there come a movie our individual construction is confronted with the directors. And this could give reasons for new constructions of the fantasy world.
Karl Popper (1972) discus the existence of a social third world of published theories. When we express a theory we give it a kind of objective existence.  We can now discus our understanding of the theory with others by referring to the printed text, and we can try to develop it or use some of the described concepts to build new theories. We can then build up a growing knowledge for the

society by critically discussions. I think it is a nice picture on the way we can share ideas and make them common - but he maybe underestimate the emotions and the politically and culturally pressure behind the discussions. For example Jerome Bruner (1996) has pointed at the narrow frames the culture sets for our constructions.

I think we in some way understand mathematical theories in the same way as we understand a novel - but the language in novels is normally more helpful. We try to build up a fantasy structure - a hypothetical world - build by some of our representations for the concept, try to connect them in a 'story' with the rules from axioms and try to relate theorems and problems to the world.

Many mathematics didactians connect representation and understanding - for example more general R. Davis and C. Maher (1990); especially drawings M. Johnsen-Hoines (1987); computer and robots S. Papert (1980). I also see structured materials as Dienes multi-base blocks and Cuisinaire blocks as tools for making representations of theories. The use of such materials follows rules like games.

If it succeed so the representations of concepts and ideas works together in a reasonable well functioned structure we can 'live' and think in this 'mathematical world' - new theorems can easily be assimilated or directly be discovered as natural consequences of the build constructions.

We go from constructing a 'world' to make discovering inside the constructed world.

Jean Piaget (1973, p. 85) writes for example:

*True understanding manifests itself by new spontaneous applications, (…): it seems that the subject has been able to discover for himself the true reasons involved in the understanding of a situation and, therefore, has at least partially re-invented it for himself.*

- But of course - sometimes the different representations of concepts do not fit together so our understanding is badly connected.

Sub-conclusion: Students books should try to build concepts up in a way, so they fit together. Try to offer a connecting 'story'.

<u>*Possible conceptual conflict in the extension of numbers:*</u>

*We will start to look at the extensions from the theoretically deductionally angle.*

The numbers are often understood as build by extensions from the natural numbers:
In school-language: So the difference always exists. So we can divide with all numbers except zero. So square root of 2 is a number.  (The next step is the complex numbers - but I will stop with the real numbers - as normal for the general school-education.)
From an axiomatical structurally point of view the extensions are relatively 'smooth': We start with the natural numbers (N, +, *) structured as commutative semi-groups both in relation to addition and multiplication. And the distributive laws connect the two operations. With the extension to the integers (Z, +, *) the semigroup (N, +) 'grows' to an Abelian group. With the extension to the rational numbers (Q, +, ') the semigroup (Z \ {0}, *) also 'grows' to an Abelian group, and we get a field. And with the extension to the real numbers (R, +, *) we still have a field.

The conceptual changes are larger in relation to order. But the existence of an order relation there is totally and working harmonic together with addition and multiplication is common for N, Z, Q and R.  (This is first changed with the complex numbers.)
The changes can be understood, as we extend with still more numbers.  We then go from a well ordered to a descrete ordered, to a dense and finally a Dedekind-complete ordered set.

The educational problem with an axiomatic description is that it describes the inside relations - the structure in a 'pure form'.  We need for our understanding something in our mind to referring to - and the society need numerals / number symbols for practical communication and description.
Also many mathematicians want 'something' to relate to theories. Some of the discussion between Frege and Dedekind is about that question (Tait, 1992). Frege and Russel try to point out some kind of defined objects as numbers. Where the extension of equinumerous / equipollent gives a kind of natural basis for natural numbers as finite cardinal numbers, the steps further for Z, Q and R are more arbitrary -for example (Russell 1971, p 64):
*Generally, if m is any inductive number, +m will be the relation of n+m to n (for any n), and -m will be the relation of n to n+m.*
And the result is a 'divided number concept' (Ibid, p 63):
*'One of the mistakes that have delayed the discovery of correct definitions in this region is the common idea that each extension of number included the previous sorts as special cases.'*

Why intuitively prefer a relation (n+m, n) for '+m' instead of relation (n, n+m) or just the sign '+m'?

Should we say that +3 = +3 is false if they represent different number objects as '(n, n+m)' and '(n+m, n)'? Or should we try to establish an isomorphism? And is then +3 = -3 because of the isomorphism between the related number-objects behind?

The ideas are nearly related to one way to introduce fractions by constructing a whole new world of symbols - so we have to translate old numbers as 3 to the new symbols as 3/1 or 6/2 etc. From a theoretical point of view this is very convincing because the rules for the new structure can be deduced from the old structure. But the cost is some conceptual breaks. If we merely look at the process as an extension of language we can diminish the breaks - and they can be enlarge if we postulate that behind the symbols we have different kind of existent number object.

*Number and the social world:*

If we look at the history of numbers then negative numbers should be some of the breaking points. But Lehtinen did not speak about the negative numbers as a hard problem for the Finnish students - so why is it not a hard problem to day?

Primarily we live in a culture, where negative numbers are used in our daily life. In relating to accounts - for example from banks. We have 'in- and output' and the sum as the balance. This can be illustrated graphically for example related to scales, number lines and co-ordinate systems. Secondly we have easily learned number symbols for the negative numbers: '-a' for the opposite number to 'a'. Thirdly negative numbers is easily to order by the 'opposite to a' - especially in relation to number lines. Finally there are not so much new to learn before we can calculate with negative numbers. We normally know about subtraction of positive numbers before negative numbers are introduced, and by the equation a + (- b) = a - b we can relate addition of (-b) with subtraction of b. Normally only the product of two negative numbers is difficult to understand for students. They often try with something like debt * debt? We here need a good 'story' to represent the problem. A possibility could be to use movements along number lines - see later.

Sub-conclusion: How students can grab a concept may be seen in relation to the actual culture. 'The history book' is problematic as a guide - but can of course give a lot of inspiration.

*Number and the physical and social world:*

*Counting 1: Natural numbers*

Victor J. Katz write about the old Greek Mathematics (1993, p 46):
*'For the Pythagoreans, numbers always was connected with things counted. (…) In particular, for the Pythagoreans, and throughout Greek mathematics, a number meant a "multitude composed of units," that is, a counting number.'*
Piaget sees the natural numbers as a synthesis of the ordinal - and cardinal numbers - especially he connect the construction of numbers with the activities of making correspondences, classifications and seriations. (For example Beth - Piaget 1966, p 259-272)
I think this is a reasonable foundation for having some numbers - but before we can speak of the system of natural numbers, we must have a counting system - a counting-out rhyme as sounds (or) /and symbols, with that quality, that we in principle can continue for ever. Paul Lorenzen use one of the simplest system for natural numbers: │, ││, │││, …n, n│, … and shows, that we in that way can construct a basis for the natural number (A short extract in Gericke 1970, p 170-172).
It is easy for children to define addition: ││ + │││ = │││││ or multiplication: ││ * │││ = │││ + │││. They are easy to use as description for sets; they are naturally ordered and we can argue for the algebraic rules (most difficult in school is to argue for the associative law for multiplication).
We have very quickly a reasonable model for the natural numbers. So why should the children spend all that time in school? The answer is, that we do not want just a model for the natural numbers, but a special base-ten model, preferred of the society by its practical usefulness to describe this kind of numbers we typically use in our daily life and in the society.
So we have for this 'base-ten' model for N = {1, 2, 3, … , 9, 10, 11, …} an extern problems by using the model as description in relation to society and internal problems by ordering them and calculating them. Especially the school education has used a lot of effort on algorithmic ways of calculating. Luckily we now have pocket calculators and computers so we can drop lot of the rigid training of arithmetic 'standard' algorithm.

*Counting 2: Finite decimal numbers - real numbers.*
We can think of measuring the length of something. We have to choose a unit of relevant type - here typical a special chosen length we give the length 1. Then we can put one unit after the next as a ruler and count the numbers of units. If we do not think this is precisely enough we can choose a new shorter unit and continue. We can express the results as a unit 1 + b unit 2 + etc. where a, b, c are (natural) counting numbers or zero. Eventually we can make a ruler with such a system.

This system is for example used in the system of finite decimal numbers where the units are chosen so 10*unit (n+1) = unit n.  This system is normally easy to learn for students because it in a way naturally continue the base ten system - especially we can nearly use the same algorithm for the arithmetic calculations. The 'ruler' set for the number line and the ordering of the number symbols and the students are known with the symbols from the daily life.
Finite decimals are sufficiently for measuring concrete quantities, but it is naturally to understand some quantities as for example time, movements and space having a kind of continuity. We meet this strongly already in Zeno's Paradoxes and this is still the naturally point of view. (But the idea of a kind of quantification of space and time is existing in physics in relation to unite the theory for Quantum Fields and the Relativity Theory.)
If we especially measure the length of (theoretically) line segments we could think that the measuring process continue so we get the idea of infinite decimal numbers. And it is not unnaturally to move the other way too from an infinite decimal number to a line segment /point on the number line. If we earlier have connected the positive real numbers with measuring quantities (magnitudes) we so get the relation between decimal numbers, the number line and the real numbers.
The problem with infinite decimal numbers (for example 43.1824…) is that they more symbolize a possible process than they give a precise description.  Precise descriptions typically comes from algebraic problems as 'the positive solution to x * x = 5', 'the solution to 7 * x = 16' or defined through geometrically constructions as for example for trigonometric functions, or….
The number symbols for such descriptions are more indirectly - for example 'the square root of 5', '16÷7' or 'Tan $70^{o}$'. They are often difficult to place in order or to calculate. Often we therefore approximated them by finite decimal numbers if we want to order or calculate.

*Measuring understand as 'ratios'.*

We will use m(A) and m(B) as the measure of quantities A and B of same type measured by a chosen unit U.  If it is possible to choose the unit U so both m(A) and m(B) is natural numbers we says that the quantities are commensurable.  If it is impossible we say that the quantities are incommensurable.
We can use the ratio m(A) / m(B) for commensurable quantities as a general way to introduce fractions (we then often start with A as a part of B), or we can use the idea more general to introduce positive real numbers as a kind of ratio 'm(A) / m(U)' between a quantity and a chosen unit. We get a natural number when the unit measures the quantity. We get a rational number when

some part of the unit measure both the unit and the quantity, and we get a irrational number, when the unit is incommensurable with the quantity.

This ideas can be found in older schoolbooks (for example Petersen, 1879) - and earlier for example by Isaac Newton (a number understood as a ratio between like magnitudes (Tait, 1992; Gericke 1970, p 97)). The weight is then naturally in school placed on fractions, but there are a lot of conceptual problems with the use of fractions.

Primarily many normally not see 'larger' fractions (for example 2358/477 or 1181/239) as numbers, but something we should calculate. Already the old Greek looks suspicious on fraction as a number and not just a ratio between two numbers. In the society today normally only small simple fraction as 1/2, 1/3 and 1/4 is seen and treated as numbers. Secondly fraction is difficult directly to compare / 'see' the order just by examine the number symbols. Thirdly they demand new algorithmic procedures. And we also need - at least because of the use in society - the decimal numbers as before. So we get by putting the fraction in front of a school number concept a lot of conceptual problems.

So why are we still doing that? One reason is the arithmetic tradition - before the pocket calculators and the computers the use of fractions was a part of the calculation technique. Another reason is the algebraic tradition, that division expressions - especially in formulas - are written as a (kind of) fraction $\frac{a}{b}$. The problem by learning algorithmic calculation by fractions is seen as a way to learn to work in an algebraic way with expression: The rules for fractions are applied to real numbers in general.

A more directly and naturally way to learn to manipulate algebraic expression is by using algebraic rules! For example (similar to rules for subtractions): In an expression consisting of product and divisions we may:

- Place or remove 'multiplication-brackets' as we like
- Place or remove 'division-brackets' if we also change the calculation signs from multiplication to division and reverse.
- Change the order if we divide and multiply with the same numbers as before.

Then we could think like this:

$$\frac{a}{b} * \frac{c}{d} = (a \div b) * (c \div d) = a * c \div b \div d = (a * c) \div (b * d) = \frac{a*c}{b*d}$$

$$\frac{a}{b} \div \frac{c}{d} = (a \div b) \div (c \div d) = a \div b \div c * d = (a * d) \div (b * c) = \frac{a*d}{b*c}$$

And with rules like: 'a ÷ b = (a * x) ÷ (b * x)' and '(a ± b)÷ c = (a ÷ c) ± (b÷ c) we also can get addition and subtraction of 'fractions'. (Of course all with restrictions so we do not divide with zero)

Sub-conclusion: Fractions plays (of historical reasons) a too large role in modern school-education.

*'Understanding' numbers:*

Werner Blum use inspired of R v Hofe “Grundvorstellungen” to constitute meaning. As some examples on “Grundvorstellungen” (GV - “concept images”) for fractional numbers he names (Blum 200x):

*'– “part-whole“-GV (3/4 as a portion: 3 out of 4 parts)*

*– “operator“-GV (a given quantity is transformed into "3/4 of" this quantity)*

*– “ratio“-GV (3/4 as a relation between 3 parts and 4 parts)'*

One idea is that central concept images could serve for the mind as a translation tool between the formal theoretical world and the 'real world'. From the symbol 3/4 we can translate to a representation in the 'real world' and reverse. Where we normally use mathematics to makes models of 'the real world' we can also use 'the real world' to understand mathematics.

A traditional school-example: A square is divided into 100 small congruent squares - 22 with one kind of mark and 15 with another. If we have a “part-whole“-GV we can see this represent 22/100 and 15/100 or 22% and 15 %. If we also have a concept image for addition as “the union of the two portions“ we have 22% + 15% = 37% and we can in that way argue for the more general rule: a% + b% = (a+b)%.

I will call this use of the square with the small squares and with the concept images as a 'understanding model' for percent. (Model because the small square are defined as congruent and together exactly the starting square etc.)

An understanding model represents a part of a mathematical structure and can be used to introduce and 'learn' about this part of the mathematical structure. Later when we from some 'understanding models' have build a model of the mathematical structure in our mind we can use the concept images 'the other way' to 'translate' from theory and makes models of other parts of the 'real world'.

A large problem is that the concept images and the understanding models are limited in relation to the normally very generally mathematical concepts. In the example “part-whole“ typically represent fractions (percents) between 0 and 1. If we want to represent for example 2½ we could instead choose a concepts like “units and part-of-units“. For example we could use a pizza-world as

understanding model (Where all the pizza are congruent circles etc.). In this 'world' we could represent 3/4 + 1/2 etc. But it is still not so obvious to represent multiplication: (3/4 pizza) * (1/2 pizza) = ?  Then we can think of a fraction as an operation. Then we could give meaning to multiplication by 3/4 of 1/2 of 'something'  - but then addition, '3/4 of 'something' + 1/2 of 'something', gives problems, if it is not the same 'something' we relate to - etc.
If we do not carefully explain to students where the concept images and understanding models function in agreement with the theory and where they don't, we not only construct a scaffold for understanding but also for misunderstandings.

*Conclusions:*
We need the different concept image to connect the theory with possible ways to modeling the 'real world', but we need a large understanding model for numbers so we can get a connected understanding of numbers.

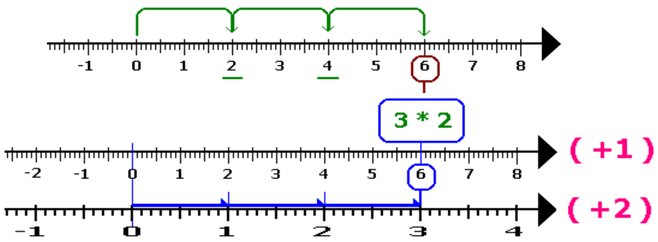
A possible operative 'school-definition' could be something like: A number is a symbol, which specify a position relative to the unit.
This build on a geometrically understanding related to a number line (or a number plane) - and the unit is understood as a kind of vector.
The concept image could be 'steps / a movement' expressed by the unit or more general a 'vector'.
The position in relation to the unit should imply the ordering of numbers.
Decimal numbers as 4.3 can be seen as directly specifying 4 'steps' of the unit followed by 3 'steps' of a shorter unit '0.1.  Fraction as 4/3 can also be seen as directly specifying 4 steps of a shorter unit '1/3' or they can be seen as more indirectly define a position relative to the unit as the result of the division '4 ÷ 3' - for example by something like '4 ÷ 3 = x ÷ 1'. Square roots etc. is normally also seen as more indirectly defined numbers.

a + b is seen as a followed by b - or more general as vector addition.
'-a' is seen as the opposite to a, directly implying that -(-a) = a.

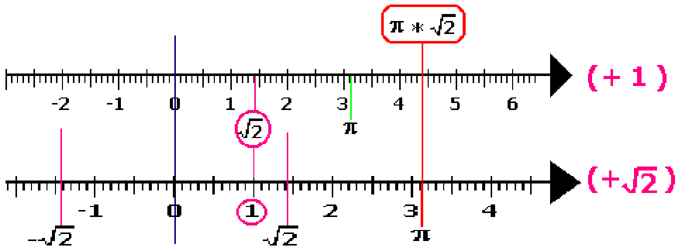
Multiplication and division can be associated with shift of unit. 'a ÷ b' is the vector a expressed in relation to b (as b now is the unit). 'a * b' is the 'new vector a' constructed from b as unit - related to the originally unit ' - Or easier - but then only for real numbers - explained by 'steps/movements' along number lines:

3 * 2 can be represented with 3 steps of length 2 units along one number line:

- Or using two number lines:




And so we also can represent irrational products:

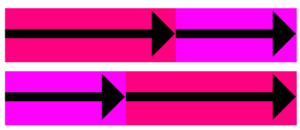



(Movements along number lines could function as a full understanding model for (R, +, -, *,÷ , <) - a more detailed version, but in Danish in Christiansen 2003.)

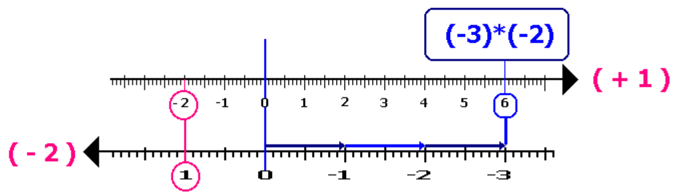
An operative 'school-definition' should - as done - reasonable could introduce order and compositions - the structure. But the connection by the concept image - 'steps or vectors' - between understanding models and theory gives also a basis for explaining the algebraic rules for numbers.
Here is shown some example on possible arguments inside a 'vector - step' context.

$(a * 2) * c \;=\; a * (2 * c)$

Half so many steps
- but each step double so long as before



a + b = b + a




A more traditional way to define numbers is to delimit the concept by using some of the 'stickers' we set on the concept as structure, language and unit. As a guiding for the schoolteacher we can in that way define numbers as:

Numbers are a structured description system based on a unit

Then the teacher has in the same definition focus on:

A structured system: compositions, order - and rules.

Description system - In the double way: - Numbers seen as symbols / words in a language, where it is naturally to give still more precisely descriptions, and therefore naturally to add new number descriptions / symbols to the earlier.
- And the use of numbers as a way to make description of the world and to communicate about it.

Based on a unit: Explaining the relation to the unit is after my opinion the foundation for understanding numbers. 'Based on a unit' is of course more vague than 'a position relative to the unit', but of the same reason possible also more easily to be accepted.

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