Addiction
in
Contemporary Mathematics

Newcomb Greenleaf
Goddard College
newcomb.greenleaf@goddard.edu

Abstract

In 1973 Errett Bishop (1928-1983) gave the title ***Schizophrenia in Contemporary Mathematics*** to his Colloquium Lectures at the summer meetings of the AMS (American Mathematical Society). It marked the end of his seven-year campaign for *Constructive* mathematics, an attempt to introduce finer distinctions into mathematical thought. Seven years earlier Bishop had been a young math superstar, able to launch his revolution with an invited address at the International

Congress of Mathematicians, a quadrennial event held in Cold War Moscow in 1966. He built on the work of L. E. J. Brouwer (1881-1966), who had discovered in 1908 that Constructive mathematics requires a more subtle logic, in which truth has a positive quality stronger than the negation of falsity. By 1973 Bishop had accepted that most of his mathematical colleagues failed to understand even the simplest Constructive mathematics, and came up with a diagnosis of *schizophrenia* for their disability. We now know, from Andrej Bauer's 2017 article, ***Five Stages of Accepting Constructive Mathematics*** that a much better diagnosis would have been *addiction*, whence our title. While habits can be kicked, schizophrenia tends to last a lifetime, which made Bishop's misdiagnosis a very serious mistake. Our account of addiction in mathematics will be illustrated in part by my own story of kicking the Classical habit. Bauer's stages are indeed those of Kübler-Ross, starting with denial and ending with acceptance. To learn to think Constructively one is likely to need to follow a path of recovery. Bauer presented a five-stage path to his audience of mathematicians, most of them thoroughly Classical. I invite you to find your own path in the thoughts that follow.

1. The Agony of Constructive Math

In 2017 I was far removed from the world of mathematics, but still a member of the American Math Society (AMS), so received in the mail the ***Bulletin of the AMS***, where I was astonished to see the title ***Five Stages of Accepting Constructive Mathematics***. When I read the first sentence, my world started to spin:

> From a psychological point of view, learning **Constructive** mathematics is **agonizing**, for it requires one to first **unlearn** certain deeply ingrained intuitions and **habits** acquired during **Classical** mathematical training. (Emphases and capitals mine.)

Not only did this sentence inform my understanding of the dialogue between Classical and Constructive thinking, but it also shed new light on significant events in my own mathematical history. And led to my writing this article.

The author is the Slovenian mathematician **Andrej Bauer**. The five stages are indeed those of Kübler-Ross: Denial, Anger, Bargaining, Depression, Acceptance.

Startling words to find in the ***AMS Bulletin***! Bauer suggests that a path to thinking Constructively is therapeutic, healing, a path of recovery. He was writing for research mathematicians, while this article is intended for others who have some connection with math but probably have never heard of Errett Bishop or his Constructive mathematics.

I anticipated some of Bauer's insight in my 1992 paper ***Bringing Mathematics Education into the Algorithmic Age***:

> Experience has shown that those who think Classically have a very difficult time understanding Constructive thought. [Classical logic] is made part of one's subconscious thinking and even questioning it seems strange.

Bauer adds the vital distinction between learning and unlearning, and the powerful insight that understanding constructive thought is difficult because it is agonizing. In the original, "Classical logic" was "The law of excluded middle", and I will explain below why I no longer use that terminology.

Bauer avoids the word "addiction", instead speaking of habits so deeply ingrained that it is agonizing to break them. I find that the concept of addiction is helpful and hopeful in this context. I kicked the Classical habit around 1970, nine years after completing my doctorate. It can be done, it's painful, but it feels great when you are free of the habit, free to think Constructively. A Constructive mathematician has no trouble thinking Classically. There is nothing to unlearn. But, depending on the mathematician and the area of mathematics, thinking Classically may or may not have much appeal.

Bauer's insight has significant implications for the education of mathematicians and others. When students are first taught to make logical arguments, it should be with a Constructive logic in which truth is positive, stronger than the negation of falsity. Then students can make a relaxed choice between Classical and Constructive ways of thinking, or both. The issue is one of freedom of choice, or freedom from addiction.

2. *Reductio* and Truth

In high school geometry I first encountered the notion of a mathematical proof. It was both thrilling and scary. How could one ever learn to make proofs? I began to feel empowered when we were given a wonderful all-purpose method, called Proof by Contradiction. In Latin this is *Reductio ad Absurdum*, and we will call it *Reductio*.

> ***Reductio.*** To prove that an assertion is true, assume it is false and derive a contradiction

Almost all mathematicians become addicted to *Reductio*, and I was no exception. A secret to mathematical success is to make logic subconscious so you can focus on the math. *Reductio* became a deeply ingrained habit that pervaded my whole world of mathematical reality at a subconscious level. Were I to question *Reductio* everything would come into question, and my whole mathematical world would crumble.

Reductio implicitly assumes that not false is the same as true, so we'll generally use the word to mean that principle.

> ***Reductio.*** If a statement is not false, then it is true: $\neg\neg A \rightarrow A$.

To evaluate this principle, let's unpack *true* and *false* Constructively.

To show that a mathematical assertion is **true** you must supply a completely convincing argument, a proof, for it. "It is true" is what you say when you are completely convinced. An argument is only completely convincing when you can convince others. Mathematical truth necessarily has a psychological component, and is often redundant. Instead of saying "A is true", simply say "A".

Classical mathematicians tend to believe that truth is pre-existing and absolute, and that logic allows them to discover it. Constructive mathematicians believe that the logic that they find convincing and use determines which statements they deem true.

To prove an assertion is **false** you must show that it implies an absurdity, an assertion that you already know is false. You need somewhere to start. As ur-falsity Bishop takes the assertion $0 = 1$, nothing equals something. Classical mathematicians commonly use the symbol $\bot$. If A is any meaningful assertion then $\neg A$ denotes the assertion that A is false. It is immediate that the assertion

$$\mathrm{A} \wedge \neg \mathrm{A}$$

is always false, and that

$$A \rightarrow \neg\neg A$$

so the converse of *Reductio* is true. But we have no procedure that converts a proof of $\neg\neg A$ into a proof of A, so *Reductio* requires a leap of faith.

3. Excluding the Middle

What kind of faith leads to acceptance of *Reductio*? Perhaps belief in a **Proof Fairy** with a magic truth wand that touches all meaningful assertions, making them either true or false. In logical notation, for any meaningful assertion A

(PF) $$A \vee \neg A$$

Then it easily follows that if A is not false, it has to be true, so Proof Fairy implies *Reductio.*

It's easy to show that the converse is also true, that *Reductio* implies Proof Fairy. Assume that

$$\neg(\mathrm{A} \vee \neg \mathrm{A})$$

Then we can deduce both $\neg\mathrm{A}$ and $\neg\neg\mathrm{A}$, which leads to a contradiction and implies

$$\neg\neg(\mathrm{A} \vee \neg \mathrm{A})$$

So Truth Fairy is not false, and we can use *Reductio* to conclude that Proof Fairy holds.

What I am calling Proof Fairy is almost universally known as **Excluded Middle**, and often attributed to Aristotle. As noted above, I formerly used the term in this way. Bauer starts his first stage, *Denial*, with this definition:

> Constructive mathematics is mathematics done without the law of excluded middle: For every proposition P, either P or not P.

Note that Bauer does not start, as I do, with Bishop's social constructivist philosophy. But for now, let's just ask: Is "excluded middle" the right name for Truth Fairy? There is room for doubt.

The middle that is excluded is any middle ground between true and false: there is no assertion that it neither true nor false:

$$\neg(\neg A \wedge \neg\neg A)$$

which is essentially the double negation of Truth Fairy, and, as we have seen, is trivially true. So in using "Excluded Middle", a negative statement that literally is trivially true, for the positive assertion that I call Truth Fairy, I submit that we are giving in to the addiction to *Reductio*. It is another habit that I have kicked.

So, Excluded Middle is the double negation of Truth Fairy, and is Constructively true.

We will show below that any implication implies its contrapositive, and from that it follows that triple negation is equivalent to single negation: $\neg A \leftrightarrow \neg\neg\neg A$, which means that $\neg\neg A \leftrightarrow \neg\neg\neg\neg A$, etc.

4. Brouwer Seen from Princeton

By the time I reached graduate school at Princeton, the habits of mind based on *Reductio* were deeply embedded in my thinking. It was there, in 1961, that I experienced Brouwer's only visit to America. I was familiar with him as one of the great pioneers of topology, by then already a vast branch of mathematics. But Brouwer was talking about the Constructive math that he called Intuitionism. No one at Princeton had any interest—indeed we can now see that it was a painful topic—and he was not invited to speak.

Still, Brouwer's presence on the continent had a large impact. The mood in Princeton's Math Common Room changed dramatically. For days on end attention was focused on Brouwer, and loud hilarity prevailed. I can now see that humor masked the pain we felt when we tried to understand.

The man was touring the world with a counter-example to his own greatest theorem! What could be crazier? So funny, but so sad too. A great mathematician gone bad. We invented a history in which Brouwer had been a young mathematician when he helped to lay the foundations of Topology. Then in old age he had started to worry about philosophy and stopped making sense. A cautionary tale about dangerous ideas. We attributed to senility the discoveries that Brouwer had actually made as a graduate student. This fabrication was widespread among mathematicians, and I later learned from colleagues that the hilarity and false history of Princeton also took place in other leading mathematics departments, including Harvard.

Now Brouwer's biography is well known, and it is clear that after he had his Constructive insight in 1908, he hid his Constructive views until he was widely recognized as a Classical topologist. After World War I he came out of the Constructive closet and created quite a stir. The leading mathematician of the day, David Hilbert, at first was impressed with Brouwer's ideas, but soon changed his mind, writing in 1927:

> Taking [*Reductio*] from the mathematician would be the same, say, as proscribing the telescope to the astronomer or to the boxer the use of his fists. For, compared with the immense expanse of modern mathematics, what would the wretched remnants mean, the few isolated results, incomplete and unrelated, that the Intuitionists have obtained.

The fists of the boxer suggest that Hilbert felt under painful attack. A Freudian might see the telescope of the astronomer as a phallic symbol that suggests a fear of emasculation. He was clearly feeling the agony that Bauer predicts. But Hilbert was right about "wretched remnants … few isolated results". Other mathematicians who tried to follow Brouwer also found the going painful and were all too eager to accept Hilbert's verdict. Not until Bishop in 1966 was it possible to envisage a Constructivization of "the immense expanse of modern mathematics". (Instead of "*Reductio*", Hilbert wrote "excluded middle".)

Brouwer's great achievement was to find a logic that worked for Constructive math. He called his philosophy Intuitionism and the logic is known as *Intuitionist Logic*, but I will use the more descriptive name **Constructive Logic**. We will take a look at it below in the beautiful Natural Deduction format that Gerhard Gentzen developed for it in the 1930s. Natural Deduction defines the logical connectives,

and ($\wedge$) *implies* ($\rightarrow$) *or* ($\vee$) *not* ($\neg$)

by rules for their *introduction* and *elimination*. It leads to proofs in the form of proof trees, with assumptions at the top and conclusion below. In contrast, the truth table definitions of Classical logic do not connect nearly so directly with proofs.

5. Errett Bishop

1966 was a significant year in the history of Constructive math. Brouwer died at age 85, with the Constructive math that he called Intuitionism barely surviving even in the Netherlands. Also in 1966, the International Congress of Mathematicians was held in Moscow, with headlines reporting the protests against the Vietnam war by the American mathematician Stephen Smale. While Errett Bishop proposed a revolution in mathematical thought, he got no headlines.

When Bishop, a young math superstar, was invited to give a major address, it was assumed that he would talk about his groundbreaking Classical research in areas like several complex variables or uniform algebras. Instead, he gave a relatively elementary talk, *A Constructivization of Abstract Mathematical Analysis,* whose title hid its revolutionary intent.

I was then an Assistant Professor of Math at the University of Rochester, where one of my colleagues was Stanley Tennenbaum, who was friends with Bishop from the University of Chicago, where they had both been graduate students. Stan was in awe of Bishop's mind, and very eager to learn what he was up to. When Bishop's **Foundations of Constructive Analysis** appeared in 1967 we both began to read it. But I had a hard time, even though it seemed simple. I would work on it for a while and think I had learned something, and then it would stop making sense. Post Bauer, I can see that I often felt anxiety or depression during the experience. While I didn't get it, I developed a motivation which led in three years to recovery from my addiction to *Reductio*.

I did learn that a change in speech was needed. Bishop started from social constructivism, the belief that mathematical objects only exist when they have been socially constructed. Mathematics is centered on a collection of agreements about what must be done to construct objects. This suggests that instead of asking

What is an integer? or What is a set?

we should ask

What must be done to construct an integer?

or

What must be done to construct a set?

Integers, sets, and real numbers will be our principal examples of constructions.

What must be done to construct a positive integer? We know integers through their representation in the standard decimal notation now known as Hindu-Arabic. If you write a number in that notation, you have certainly constructed it. In ***Schizophrenia***, Bishop started from what he called

> **The Fundamental Constructivist Thesis**: Every integer can be converted to decimal form by a finite, purely routine, process.

To construct an integer, you must provide an algorithm which will compute it in decimal form. The algorithm is usually so obvious that it is not stated. "The number of primes less than a billion" is accepted as a complete construction of an integer, which can be computed with a simple computer program. But, as the example of "the number of primes less than 9^{9^9}" shows, it may be naïve to start running the algorithm and wait for the result, no matter how fast your computer.

6. Brouwer Encodes Ignorance

In 1908, when still a graduate student in Amsterdam, L. E. J. Brouwer published an obscure paper in which he showed that *Reductio* is incompatible with Bishop's Fundamental Constructivist Thesis. With *Reductio* he could construct non-computable integers. Because he was a graduate student, nobody paid any attention. By 1920 Brouwer was recognized as a leading researcher in the new mathematical field of topology, where the Brouwer Fixed-point Theorem is a foundational result, and could no longer be ignored.

Brouwer's construction of a non-computable integer is extremely simple, but you may find it painful to contemplate. He started with the observation that mathematical ignorance is vast: it's easy to state assertions that we have no hope of proving or disproving. He was fond of assertions about the decimal expansion of π. Computer scientists have found very efficient algorithms for computing π. "Many trillions" of digits have now been computed, according to Wikipedia. While this is a very large number, it is finite and there are still infinitely many digits to come, so in that sense we've barely started. The digits appear rather random, so there is no current hope of proving or disproving assertions like:

The decimal expansion of π contains a sequence of 100 consecutive 7's.

Nothing special about 100 consecutive 7's. You can make up your own finite sequence of digits and your own hopeless assertion. The length of the sequence depends on one's computing power. Brouwer wrote of ten consecutive 7's and Davies uses a thousand. I feel a hundred is enough for now.

Now for the "non-computable integer". Call your favorite hopeless assertion H, and define the mathematical object B (for Brouwer) as follows

$$B = \begin{cases} 0 & if\ H\ is\ false \\ 1 & if\ H\ is\ true \end{cases}$$

That's the whole construction. The only way to compute B is to either prove or disprove H, which we have no hope of doing.

While we cannot prove the assertion "B is an integer", we can easily show that it is not false. For if B is not an integer, then it is neither 0 nor 1, which means that H is both false and not false which is a contradiction. So if we accept *Reductio*, then it becomes true that B is an integer, and we are forced to admit non-computable integers. We will see that B is actually an interesting example of a Constructive set.

To keep integers constructive, we must reject *Reductio* and take truth as stronger than the negation of falsity. Truth must have a positive quality. Classical mathematicians, ruled by their addiction to *Reductio*, have chosen to give up on integers being computable.

7. Constructive Sets

Classical mathematicians have a material understanding of sets. A set *consists* of its elements somewhat as a brick consists of its atoms. The elements exist first, and subsequently are collected into a set. With this understanding, it would seem impossible to construct infinite sets. Bishop's constructive approach reverses the priority of set and element. Of course the number 7 was in wide use long before anyone conceived of the infinite set $\mathbb{N}$ of all positive integers, but 7 couldn't exist *as an element of that set*, until $\mathbb{N}$ itself was constructed, a construction we will repeat in a moment. The construction of a set creates the possibility of constructing elements of the set.

Bishop's Definition. To construct a **set** S you must do three things.

- Describe what must be done to construct an arbitrary element of S.
- Describe what must be done to prove that two arbitrary elements of S are equal.
- Prove that your equality relation has the usual properties: reflexive, symmetric, transitive.

Step 1 for the set $\mathbb{N}$ of positive integers is the Fundamental Constructivist Thesis. Here is Step 2: To prove that two integers are equal, you must prove that their standard decimal representations are identical. Step 3 follows because identity of symbols is reflexive, symmetric, and transitive. That completes the construction of the infinite set $\mathbb{N}$.

Bishop liked to emphasize that equality is always a convention. There is no underlying relation of identity. Given two arbitrary mathematical constructions, it makes no sense to ask if they are the "same". It makes sense to ask if they are equal if both are elements of the same set. What then are we to make of the reflexive law: $x = x$. It seems to say that if two elements are identical, then the equality relation will call them equal. The constructive understanding does not need the identity relation, but instead relies on the fact that mathematical constructions, like scientific experiments, are intentional acts that are **repeatable**.

The reflexive law states that if you make a construction and then repeat it, the original equals the copy. It may be worth noting that "role a die and count the pips" does not count as the construction of an integer, because if you roll a die twice, the results may not be equal.

Now we can understand Brouwer's "non-computable integer" B as a subset of the integers. To construct an element of B either prove H and construct 1, or prove *not* H and construct 0. While B is non-empty (it is false that B is empty), we have no routine way to construct an element of B. Constructively, there is a difference between *non-empty* and *inhabited*. It is also easy to show that any two elements of B are equal.

8. Kicking the Classical Habit

Back in 1967 Rochester, Stan Tennenbaum and I tried to read Bishop's book together, but it didn't work, because Stan was getting it, and I wasn't. Now I can see the pain, the anxiety and depression, that took me down when I tried to learn. But Stan's example convinced me that I wanted to understand. There was another issue. I was an untenured faculty member, and I could see how Stan's conversion to Constructive thinking made our colleagues uneasy, an observation that probably also had its impact on my motivation.

A year later both Stan and I left Rochester. I became a tenured Associate Professor at Texas, while Stan spent a semester at New Mexico State, where he gave a seminar on Bishop's Constructive math. That seminar created a very productive research group in Constructive math at NMSU, which lasted for many years into the present. It was a totally unique event, since those who actually got Bishop were otherwise very few and very far between. Fred Richman became, and remains, the most productive member of that group, although he was on leave when Stan was there.

In 1969 I attended the Hedrick Lectures that Bishop gave for the Mathematical Association of America. As he started to speak, there would be lots of nodding and smiling, for he was a clear and engaging expositor. But as he went on, our nodding would gradually stop and our smiles would turn to frowns, as we realized that it just wasn't making sense. Bishop could feel that he was losing his audience, and couldn't understand why.

At the Hedrick Lectures I reconnected with Gabriel Stolzenberg, with whom I had shared an office when we were Peirce Instructors at Harvard. Like Stan, Gabe was a friend of Errett and was among the few who had recovered from addiction to *Reductio*. His review of Errett's book was soon to appear in the 1970 **AMS Bulletin**. I told him of my frustration, and he volunteered to help me learn to think constructively. We started with me reading his review. What stands out when I re-read it now is:

> [Bishop] is not joking when he suggests that classical mathematics, as presently practiced, will probably cease to exist as an independent discipline once the implications and advantages of the constructivist program are realized. … I fully agree with this prediction.

Both Errett and Gabe were blind to the addictive force of *Reductio*.

I was in Texas and Gabe was in Cambridge and long-distance calls were expensive, so over a period of many months we had a regular snail-mail correspondence about Constructive math. Unfortunately, most of it has been lost. Somehow, Gabe led me back to a place where I felt that I had a choice. That was the unlearning. From then on it was relatively easy. I did a piece of Constructive math that is my favorite of all my own research: ***Linear Order in Lattices: a Constructive Study***, published in 1978. It would make a nice topic for an undergraduate class in Constructive algebra if such courses existed.

While I was safely tenured, I did not find it easy being the only Constructivist in a huge department in which most, perhaps all, of my colleagues were addicted to Classical thinking, as I had been. I was personally popular but professionally painful. The curriculum for math majors and graduate students was thoroughly Classical, and while I could teach it, I didn't particularly want to. I had a few graduate students who got into Constructive thinking and even supervised the Constructive Ph. D. of Pearl Olson, who didn't even try to get a teaching job in a math department. My sense of alienation increased and led to my resignation from tenure in 1976.

9. Schizophrenia

The stories of Brouwer and Bishop have a common core that Bauer's analysis illuminates. Both were recognized as world-class researchers in Classical mathematics when they proposed the advantages of a Constructive approach. Both got a lot of attention at first, and were then astonished that so few mathematicians understood. Neither realized the significance of the difference between learning and unlearning, nor understood of the pain caused by their ideas, the addiction to Classical thinking.

Bishop's reputation as a world-class researcher persisted and in 1973 he was invited to give the Colloquium Lectures at the summer meetings of the AMS. He knew that he was not being understood, and decided that it was because Classical mathematicians were mentally ill and that he was being too polite about it, and gave his talks the title ***Schizophrenia in Contemporary Mathematics***. The notes that were passed out for those talks are an excellent introduction to his thought, much of it accessible for an audience beyond the AMS.

The term "Schizophrenia" was offensive and unhelpful, especially as it implied an incurable condition. The reception of the audience became tinged with active anger. As we noted at the start, Bauer agrees that Classical mathematicians have a mental problem, but his analysis would suggest that "schizophrenia" be replaced by "addiction".

After the Colloquium Lectures Bishop seemed to give up. He stopped taking research students, did not publish the notes to ***Schizophrenia*** or any further mathematics, and died of cancer 10 years later. A new edition of his (1967) book, simply called **Constructive Analysis** was completed after his death by Douglas Bridges. In 1985, the AMS published a memorial volume **Errett Bishop: Reflections on Him and His Research** that included the notes for ***Schizophrenia***. Alas, they are virtually unknown either in math or math education. Bauer does not mention them, and they are not in the bibliography of Paul Ernest's **Social Constructivism as a Philosophy of Mathematics**. But the memorial volume is still available from the AMS, and ***Schizophrenia*** can now be found online.

There are several reasons why ***Schizophrenia*** vanished almost without a trace. It appeared in Volume 39 in the somewhat obscure AMS series **Contemporary Mathematics**. The word "schizophrenia" was offensive and misleading, and, as Bauer has taught us, Constructive math is painful stuff. Beyond that, in addition to ***Schizophrenia***, the volume contained the proceedings of an academic conference organized by Bishop's mathematics department at University of California San Diego, and the karma of that conference was very bad.

From the Preface: "a number of his former colleagues were invited to speak about him and his work". Those colleagues were mostly famous Classical mathematicians, like John Kelley, John Wermer, and Irving Glicksburg, who had collaborated with him before he turned to Constructive math, of which they had limited appreciation and understanding. The only Constructivist invited was my mentor Gabriel Stolzenberg. From the Preface: "Unfortunately, Stolzenberg withdrew his paper on Errett Bishop." Gabriel found it almost obscene that a meeting in Bishop's memory should largely focus on his early Classical research, and almost ignore his Constructive period, the culmination of his mathematical career. Gabe stormed off at some point and withdrew his paper, which was never published. He had given me a copy before the conference, but at some point it disappeared. Gabriel died earlier this year. I asked his widow if a copy still existed. Apparently not.

10. Constructivity and Algorithms

After leaving Texas, I spent some time in Boulder, studying and practicing with the Tibetan Buddhist meditation master Chögyam Trungpa. There a friend suggested that I work in computers where he did. I had never even seen a computer (it was the 1970s), but he assured me that, as a mathematician, I would easily learn on the job. That turned out to be right, and I spent several years working and learning at Sigma Design, a small computer graphics company in Denver. It helped that I had learned to think Constructively, since Constructive mathematics has a deep affinity with computer science.

Bishop realized early on that Constructive mathematics could be seen as a very high-level programming language, see (1970), but tended to downplay that realization because he feared that mathematicians would classify Constructive mathematics as computer science. For the same reason, he avoided using the word algorithm, preferring phrases like *finite routine procedure*. In retrospect, if Bishop had fully understood the extent of addiction to *Reductio* among mathematicians, he might have embraced algorithms and moved to a CS department to found Constructive Mathematics as an alternative discipline that linked CS with mathematics. It would have made things interesting.

In the 1980s, Stanford's great computer science pioneer Donald Knuth wrote of Bishop's book:

> The interesting thing about this book is that it reads essentially like ordinary mathematics, yet it is entirely algorithmic in nature if you look between the lines.

His review ***Algorithmic Thinking and Mathematical Thinking*** appeared first for computer scientists in 1981, and was later revised for mathematicians in 1985. In particular, Knuth pointed out that a Constructive mathematician can understand the triad

(Assumption Proof Conclusion)

in terms of

(Input data Algorithm Output data)

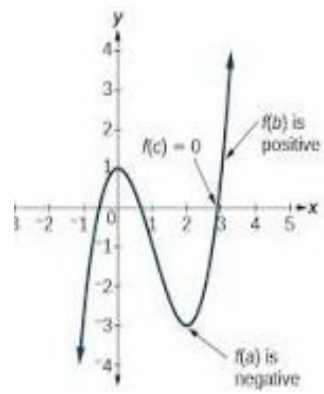
Understanding proofs as algorithms can make the concept of a proof more accessible for many students. Knuth seemed puzzled that mathematicians did not make Constructive math a regular part of their curriculum. Bauer makes it clear why that didn't happen.

Constructive (Intuitionist) Logic relies on algorithms. A proof that A implies B is an algorithm that converts any possible proof of A into a proof of B. This is the main place that algorithms enter into the Constructive logic of propositions. Almost everything else is covered by the introduction and elimination rules of Natural Deduction which are given below.

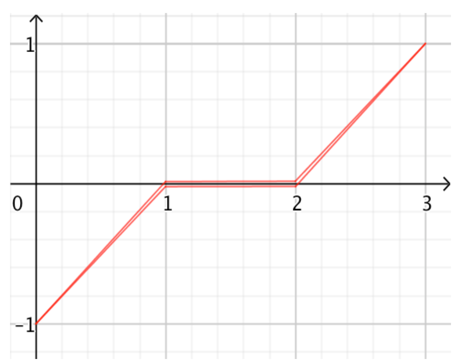
In 1986 I returned to academia as a Constructive mathematician in the Computer Science department of Columbia. It was a time of acute shortage of Ph. D.s in CS, and they were happy to have a Constructive mathematician teach basic programming and some of their theoretical courses. I stayed for 8 years as an untenured full Professor until the tenure system ejected me. It was a wonderful experience in total contrast to Texas. Computer scientists find Constructive math natural and pain free. Nobody was addicted to *Reductio*. It was again fun to talk math.

11. The Intermediate Value Theorem

Let's return to the counter-example that Brouwer was presenting when I was at Princeton, the counter-example to his own greatest theorem that we found so hilarious. I learned years later that the counterexample was really to the one-dimensional case of his theorem, which is equivalent to the Intermediate Value Theorem of elementary calculus: a continuous function that goes from negative to positive takes the value zero somewhere in between. When you see a graph like this, the theorem seems obvious. But, as Bishop would point out, this is a "dumb" function.



A "smart" function S would go up to the horizontal axis and then hover close to it for a while, so you might not know where it actually crossed. Brouwer showed how to construct such a smart function. It also involves encoding ignorance, this time in a real number, and with no use of *Reductio*. In the next section we will construct a real number U whose sign is unknown. The number U is not known to be positive, negative, or zero, though we know that it is very, very close to zero. We know that it is not irrational, but can't prove that it is rational. Our smart function S is constant with value U between 1 and 2, and is extended continuously as shown. There are three possibilities for the set of values x with $S(x) = 0$:



- All x between 1 and 2.

- A unique x slightly less than 1.
- A unique x slightly greater than 2.

Since we cannot rule out any of the three possibilities, we are unable to begin to locate a value x_0 with $S(x_0) = 0$. The Intermediate Value Theorem cannot be proved Constructively. A Constructive Intermediate Value Theorem needs an additional assumption which rules out examples like this. For example, the Intermediate Value Theorem can easily be proved if the function is *strictly* increasing, if $a < b$ implies that $S(a) < S(b)$.

12. Real Numbers and Complexity

Now let's look at the Constructive real numbers, and then we will construct a real number U which we are unable to show to be positive, negative, or zero. To construct a **rational number** you must construct two integers, an arbitrary numerator and a non-zero denominator. The Constructive arithmetic of rational numbers holds no surprises.

A **real number** is an algorithm that generates a convergent sequence $\{c_n\}$ of rational numbers. There are various equivalent ways of defining what it means for a sequence to converge. A simple definition is to say that the n^{th} term in the sequence must be within $1/n$ of all subsequent terms:

$$|c_n - c_{n+k}| \leq 1/n$$

for all positive integers k and n.

How is equality defined? Real numbers $\{c_n\}$ and $\{d_n\}$ are *equal* if for all n

$$|c_n - d_n| \leq 2/n$$

It is not totally obvious that this relationship is transitive. It is a good exercise in Constructive thinking, or check it out in Bishop's work.

We will now construct a real number U such that we have no way of deciding if it is positive, negative, or zero. It is again a question of encoding ignorance, but this time in a real number (and with no use of *Reductio*). This algorithm for U uses an algorithm for generating the decimal expansion of π. Start generating π and keep close watch for a sequence of 100 consecutive 7's. As long as you haven't found such a sequence, set $U_n = 0$. But suppose at the m-th place you first see 100 consecutive 7's. Then stop the presses. For all $n \geq m$, set

$$U_n = \begin{cases} \frac{1}{m} & \text{if } m \text{ is even} \\ -\frac{1}{m} & \text{if } m \text{ is odd} \end{cases}$$

Our ignorance of the decimal expansion of π has been encoded into the sign of U. We don't know if U equals 0, is positive, or is negative. It is easy to show that U is not irrational but we cannot show that it is rational. As we have shown, the value U can be used to construct a continuous function that goes from negative to positive, yet we are unable to find a place where it is zero.

We cannot decide whether our number U equals 0. Equality of real numbers is not decidable. The situation is very different for integers and for rational numbers. Given two integers, we can compare their decimal representations and either find a difference or confirm that they are the same. Equality of rational numbers reduces to equality of integers by cross-multiplication and is also decidable.

Classical mathematicians have trouble understanding the set $\mathbb{R}$ of Constructive real numbers because it seems to be both countable and uncountable. The Cantor diagonal argument—an algorithm that, given a sequence of real numbers, produces a real number different from every number in the sequence—is completely Constructive, and seems to show that the set is uncountable. But every real number is given by an algorithm that is described by a finite sequence of symbols, and the set of all finite sequences of symbols is countable.

The situation clarifies if we see that Cantor discovered a difference in *complexity* rather than in size. The set $\mathbb{R}$ is not *bigger*, but *more complex* than the set $\mathbb{N}$. Its complexity is related to the fact that real numbers are algorithms, and to the undecidability of the halting problem shown by Turing. Given a set of symbols purporting to describe the algorithm for a real number, Turing showed that we have no algorithm that decides whether it actually computes a real number or goes into an infinite loop. So we have no way to make a list of all real numbers.

13. Natural Deduction

This section is for those who want a more concrete understanding of the Constructive logic that Brouwer developed. He did not give explicit axioms for it, leaving that to his student Arend Heyting who did so in 1930. Heyting's axioms were technical looking and unintuitive. But a few years later Gerhard Gentzen showed that Constructive Logic could be given a natural and elegant definition in the Natural Deduction format of rules for the *introduction* and *elimination* of the logical connectives. Before presenting the rules for the connectives, I feel compelled to say a word about Gentzen, who should be remembered more for the extraordinary way he enriched our understanding of logic than for his tragic death at age 36 in 1945. He and other German nationals were starved while imprisoned by the Soviet army in Prague at the end of the World War.

Bishop never presented the Natural Deduction format. Just as he avoided the word "algorithm" to avoid being considered a computer scientist, he avoided paying too much attention to logic to avoid being considered a philosopher.

Arguments in Natural Deduction are naturally given as downward growing trees, with assumptions at the top, conclusion at the root, with the introduction and elimination rules acting in between. Here are the rules for the connectives:

and (∧) *implies* (→) *or* (∨) *not* (¬)

The connective *and* holds no surprises. There is one introduction rule

$$\frac{A \qquad B}{A \wedge B}$$

and two elimination rules:

$$\frac{A \wedge B}{A} \qquad \frac{A \wedge B}{B}$$

As we have indicated, to show that A *implies* B, we must provide a method that converts any possible proof of A into a proof of B. In Natural Deduction this is represented by a proof tree that has A at the top and B at the bottom. When we conclude that $A \to B$, this is no longer under the assumption that A is true. In the proof tree this is represented by "bracketing" the assumption A when deducing the conclusion $A \rightarrow B$. We have proved $A \rightarrow B$, not that A implies $A \rightarrow B$.

$$\frac{\begin{array}{c}[A]\\ \cdots\\ \cdots\\ \cdots\\ B\end{array}}{A \rightarrow B}$$

The elimination rule for *implies* is just *modus ponens*.

$$\frac{A \quad A \rightarrow B}{B}$$

There are two introduction rules for *or*:

$$\frac{A}{A \vee B} \qquad \frac{B}{A \vee B}$$

The elimination rule for *or* is Proof by Cases:

$$\frac{A \vee B \quad A \rightarrow C \quad B \rightarrow C}{C}$$

The introduction rule for *not* follows from the introduction rule for implication, since to prove *not A* you must show that *A* implies a contradiction, say 0 = 1.

$$\frac{\begin{array}{c}[A]\\ \cdots\\ \cdots\\ \cdots\\ 0 = 1\end{array}}{\neg A}$$

The assumption *A* is bracketed when we conclude $\neg A$.

Elimination rule for *not* can be seen as proof by cases, but can also be understood directly.

$$\frac{A \vee B \qquad \neg A}{B}$$

If you have a proof of ¬A, then you can't have a proof of A, so it you have a proof of A or a proof of B, it must be a proof of B.

Here is an example of a proof tree. It shows that an implication implies its contrapositive.

$$\dfrac{\dfrac{\dfrac{\dfrac{[A] \qquad A \rightarrow B}{B} \qquad [\neg B]}{0 = 1}}{\neg A}}{\neg B \rightarrow \neg A}$$

This allows us to see why ¬A and ¬¬¬A are equivalent. Since A implies ¬¬A, then the contrapositive holds: ¬¬¬A → ¬A. Since ¬¬¬A is the double negation of ¬A, ¬A → ¬¬¬A, and we're done.

14. Mathematical Pluralism

Most mathematicians, be we Constructive, Classical, or other, are monists. We believe there is only one true mathematics, which we just happen to practice. In two 2005 papers about "mathematical pluralism", E. Brian Davies suggested a different view:

> We approach the philosophy of mathematics via a discussion of the differences between Classical mathematics and Constructive mathematics, arguing that each is a valid activity within its own context.

Bauer would predict that Classical mathematicians would avoid such a discussion like the plague. Davies never returned to the topic and I suspect that it was because of almost total lack of response. He remained prolific for a decade.

It is fine to accept Constructive and Classical as equally valid, but that does not make their relationship symmetric. As Bauer points out, it is most often agonizing for Classical mathematicians to try to think Constructively. The example of David Hilbert suggests that this may be particularly true for great mathematicians. But a Constructive mathematician has no difficulty thinking Classically. Classical mathematics studies the consequences of *Reductio*. There is nothing to unlearn.

Mathematical Pluralism emphasizes the issue of Freedom of Choice. For mathematicians, a choice between research done Classically or done Constructively. For them and for others who use or enjoy math, a choice between Classical and Constructive understanding, which can influence how we understand other aspects of our world. Does Constructive mathematics encourage you to think of truth as having positive qualities in other areas of your life?

To make such choices genuinely available will require intervening in math education to ensure that Constructive thinking is taught when students are first introduced to proving theorems. Otherwise Constructive thinking becomes too painful to be a genuine choice.

It is not just those with social Constructivist views who are ill-served. Even confirmed Platonists who believe deeply in the Truth Fairy might find that the more subtle distinctions of Constructive logic open up new understandings. Davies pointed out one major source of such insights:

> Classical mathematics may provide a more flexible context for proving the existence of entities, but constructive mathematics provides a systematic approach to understanding why computational solutions of problems are sometimes not easy to obtain.

Mathematical pluralism is supported in a much broader sense by the field of ethnomathematics, which has shown that a wide variety of mathematical thinking and communication can be found across the range of human cultures and civilizations.

15. Addiction in History of Math

We close by noting that addiction to *Reductio* is not confined to mathematics. It can also be seen in the history of math. You may recall that ***primes*** are positive integers like 2, 3, 5, 7, 11, 13, … that cannot be factored. In Euclid's **Elements** there is a proof that there are infinitely many primes.

As a math student I was taught on several occasions that Euclid started with the assumption that there are only finitely many primes, and then logically derived a contradiction, proving there must be infinitely many. Renowned scholars have made this claim. It has been recited in myriad classrooms. As a Classical mathematician, I bought it and taught it.

Actually, Euclid himself did no such thing. The myth apparently started when some translator compulsively added *Reductio*, and was compulsively copied. Euclid merely presented an algorithm that takes as input any finite set of primes, computes some prime numbers not in that finite set, and outputs them. Euclid's computation has three steps: first, multiply all input primes together, then add 1, and finally factor into primes which are output. No assumption of finiteness, no contradiction.

The false belief that Euclid engaged in a *Reductio* sideshow has proved remarkably robust, seemingly immune to correction, as fully documented in 2009 by Hardy and Woodgold. Even for historians, it appears painful to doubt *Reductio*.

References

Andrej Bauer (2017) *Five Stages of Accepting Constructive Mathematics*, ***Bulletin of the American Mathematical Society***

Errett Bishop (1966) *A Constructivization of Abstract Mathematical Analysis*, ***Proceedings of the 1966 International Congress of Mathematicians***

Errett Bishop (1967) **Foundations of Constructive Analysis**, McGraw-Hill

Errett Bishop (1970) *Mathematics as a Numerical Language*, **Intuitionism and Proof Theory**, North-Holland

Errett Bishop (1973-1985) *Schizophrenia in Contemporary Mathematics*. Notes for the ***Colloquium Lectures*** distributed at the American Mathematical Society summer meetings in 1973. Published posthumously in 1985 in **Errett Bishop: Reflections on Him and His Research,** Murray Rosenblatt, editor, ***Contemporary Mathematics*** volume 39***, American Mathematical Society***. It is still available from the AMS, and *Schizophrenia* can be found online.

Errett Bishop & Douglas Bridges (1985) **Constructive Analysis**, a revision and expansion of (1967), Springer-Verlag

L. E. J. Brouwer (1908) *The Unreliability of the Logical Principles*, **Collected Works Volume 1**, North-Holland (1975)

E. Brian Davies (2005a) *Pluralism in Mathematics*, ***Philosophical Transactions-Royal Society of London***

E. Brian Davies (2005b) *A Defence of Mathematical Pluralism*, ***Philosophia Mathematica***

Newcomb Greenleaf (1978) *Linear Order in Lattices: A Constructive Study*, **Studies in Foundations and Combinatorics**, Gian-Carlo Rota, editor, Academic Press

Newcomb Greenleaf (1980) *Liberal Constructive Set Theory*, in **Constructive Mathematics**, Fred Richman, editor, ***Lecture Notes in Mathematics*** #873, Springer-Verlag

Newcomb Greenleaf (1989a) *Algorithms and Proofs: Mathematics in the Computing Curriculum*, ***Association of Computing Machinery (ACM) SIGCSE Bulletin***

Newcomb Greenleaf, Judith Wilson, and Robert Trenary (1989b) *Algorithms and Software: Integrating Theory and Practice in the Undergraduate Computer Science Curriculum*, ***ACM SIGCSE Bulletin***

Newcomb Greenleaf (1990) *Computability and data types,* ***ACM SIGCSE Bulletin***

Newcomb Greenleaf (1991) *Algorithmic Languages and the Computability of Functions*, **The Mathematical Revolution Inspired by Computing**, J. H. Johnson and M. J. Loomes, editors, Oxford University Press

Newcomb Greenleaf (1992a) *Bringing Mathematics Education into the Algorithmic Age*, **Constructivity in Computer Science, *Lecture Notes in Computer Science*** # 613, J. Paul Myers and Michael J. O'Donnell, editors, Springer-Verlag

Newcomb Greenleaf (1992b) *Algorithmics: a New Paradigm for Mathematics*, **Philosophy and the Computer**, Leslie Burkholder, editor, Westview Press

Michael Hardy and Catherine Woodgold (2009) *Prime Simplicity*, ***Mathematical Intelligencer***

Donald E. Knuth (1981) *Algorithmic Thinking and Mathematical Thinking*, **Algorithms in Modern Mathematics and Computer Science,** D. E. Knuth and A. P. Ershov editors, ***Lecture Notes in Computer Science*** #122, Springer-Verlag

Donald E. Knuth (1985) *Algorithmic Thinking and Mathematical Thinking,* a translation of (1981) from ALGOL to Pascal, ***The American Mathematical Monthly***

Eckhart Menzler-Trott (2007) **Logic's Lost Genius: The Life of Gerhard Gentzen**, American Mathematical Society

Gabriel Stolzenberg (1970), *Review of Errett Bishop* (1967), ***Bulletin of the American Mathematical Society***